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LETTER TO THE EDITOR

Asymmetric universal amplitude relation and its applications in two-dimensional systems

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Abstract. We have investigated the universal amplitude relation of the free energy per correlation volume $(f_s^+ - f_s^-)\xi^2$, for two-dimensional systems presenting an asymmetric behaviour in the renormalization group trajectories above (+) and below (−) the fixed point. As an example, this amplitude is exactly computed in a system possessing logarithmic specific heat singularity. We also discuss an application of our result in the renormalization group crossover of the tricritical Ising model to the Ising fixed point.

The identification of the universality class of a given physical system depends much on the universal behaviour of its thermodynamic functions in the vicinity of the critical point $\lambda = \lambda_c$. This includes not only the critical exponents but also certain critical point amplitude combinations of the scaling functions [1]. One example of such amplitude relations is the singular part of the free energy per correlation volume, $f_s\xi^d$, in a d-dimensional system [2]. The universal character of this amplitude has been shown to hold within the framework of the renormalization group (RG) approach [3]. In the specific case of two-dimensional systems, the additional principle of conformal invariance impose some restrictions on the RG scenario [5] and on the properties of the scaling functions. In fact, Cardy [4] has pointed out that it is possible to assert an exact prediction for this amplitude. Consider a RG trajectory characterized as a perturbation of the critical point by a single relevant conformal field ϕ with scaling dimension $2\Delta_\phi$. Defining the correlation length ξ as the second moment of the two-point correlation of the order parameter ϕ, his remarkable result reads

$$\lim_{t\to 0_\pm} (f_s\xi^2) = -\frac{c}{12\pi\Delta_\phi} \tag{1}$$

where $t = (\lambda - \lambda_c)/\lambda_c$ is the reduced RG coupling constant conjugated to the perturbing field ϕ, $+(-)$ refers to the regime above (below) the critical temperature λ_c and c is the central charge of the system at criticality. Due to a particular definition of ξ, we recall that (1) is only valid for $\Delta_\phi < \frac{1}{2}$.

The purpose of this letter is to discuss similar amplitude relations in systems possessing an asymmetric behaviour in the singular part of the free energy $f_s^+(f_s^-)$ for $t>0(t<0)$. By introducing a suitable definition of the scale ξ^2 our prediction of the universal amplitude $(f_s^+ - f_s^-)\xi^2$ has the quality of being valid for any relevant field $\phi(\Delta_\phi < 1)$. This includes a class of interesting and relevant physical systems. For example, we discuss a sequence of $Z(N)$ invariant models presenting a logarithmic singularity ($\Delta_\phi=\frac{1}{2}$) in the free energy. The asymmetry of the scaling free energy can be exactly estimated as well as the asymptotic behaviour of the zeros of the partition function away from criticality. We also discuss an application of our results in the crossover of the tricritical Ising to the Ising fixed point.

Our main result concerning the universal amplitude relation is obtained as follows. Let us, in the scaling region, define the square of the correlation length $\xi_\pm^2$ in terms of the following particular combination of the connected part of the order parameter ϕ two-point correlation function $\langle\phi(r)\phi(0)\rangle_c^+(\langle\phi(r)\phi(0)\rangle_c^-)$ above (below) the critical point λ_c as

$$\xi_\pm^2=\frac{\int r^2\langle\varepsilon(r)\varepsilon(0)\rangle_c^\pm\,\mathrm{d}r^2}{\int\langle\varepsilon(r)\varepsilon(0)\rangle_c^+\,\mathrm{d}r^2-\int\langle\varepsilon(r)\varepsilon(0)\rangle_c^-\,\mathrm{d}r^2}. \tag{2}$$

Then we find the following universal amplitude relation

$$\lim_{t\to 0^\pm}(f_s^+-f_s^-)\xi_\pm^2=-\frac{c-c_{\mathrm{IR}}^\pm}{12\pi\Delta_\phi} \tag{3}$$

where $c_{\mathrm{IR}}^+(c_{\mathrm{IR}}^-)$ is the central charge of the infrared fixed point (large distances behaviour) for $t>0$ ($t<0$) to which the system may cross over.

The derivation of (3) follows closely [4]. The singular part of the denominator is related to

$$-\frac{\partial^2}{\partial^2 t}(f_s^+-f_s^-)=\frac{-\Delta_\phi}{(1-\Delta_\phi)^2}t^{-2}(f_s^+-f_s^-)$$

and the numerator is estimate using the sum rule from Zamolodchikov's c-theorem [5] discussed by Cardy [4]

$$\frac{(c-c_{\mathrm{IR}}^\pm)}{12\pi(1-\Delta_\phi)^2}t^{-2}=\int r^2\langle\varepsilon(r)\varepsilon(0)\rangle_c^\pm\,\mathrm{d}r^2. \tag{4}$$

Combining these two relations we obtain the result equation (3). It has to be noted that in the denominator of (3) common singularities (for $1/2\leqslant\Delta_\phi<1$) for both signs $\pm$ are cancelled out, guaranteeing that $\xi_\pm^2$ scale with the correct critical exponent. We stress, however, that in all manipulations we have assumed a non-zero denominator in the scaling regime $|t|\to 0$, namely $f_s^+\neq f_s^-$. In the examples that will be discussed here the asymmetry of the scaling free energy is attributed to a different infrared RG behaviour ($c_{\mathrm{IR}}^+\neq c_{\mathrm{IR}}^-$). For such cases it is convenient to define a unique scale $\xi^2=\xi_+^2-\xi_-^2$ and (3) becomes,

$$\lim_{t\to 0^\pm}(f_s^+-f_s^-)\xi^2=\frac{c_{\mathrm{IR}}^+-c_{\mathrm{IR}}^-}{12\pi\Delta_\phi} \tag{5}$$

and therefore a direct relation between properties of the ultraviolet regime (left-hand side) and those from the infrared behaviour (right-hand side) has been established. A typical situation of asymmetry is when, say for $t>0$, the system crosses over to a non-trivial infrared fixed point c_{IR}^+ whereas for $t<0$ the infrared regime is governed by a massive field theory ($c_{\mathrm{IR}}^-=0$). To be more specific we are going to consider some examples below.

The first example consists of a sequence of $Z(N)$ invariant field theories in the minimal series of the WA_{N-1} algebra [6] with central charge

$$c=\frac{3(N-1)^2}{2(2N-1)}.$$

For example, $N=2$ is the Ising model and for $N=3$ ($c=1/2+7/10$) is a particular combination of critical and tricritical Ising models. The lattice version of these theories are generalized A_{N-1} RSOS models [7] in which the local states take values on the level-N dominant weights of the A_{N-1} Lie algebra [8, 9]. Perturbing the critical theory by the least $Z(N)$ relevant order parameter ϕ with conformal dimension $\Delta_\phi=1/2$, we define sequences of scaling models presenting a logarithmic singularity in the free energy $f_s^\pm$. Indeed, the conjugated 'specific heat' exponent is $\alpha=0$, and $f_s^\pm$ is expected to have the following form [10, 11],

$$f_s^\pm=A_\pm t^2+B\log|t| \qquad |t|\approx 0 \tag{6}$$

where $A_\pm$ and B are the critical amplitudes.

The advantage of these models is that even away from the criticality they are integrable and the thermodynamic Bethe ansatz approach can be applied in order to calculate the free energy [15, 16]. Using this method, and properly taking into account the singularities, the amplitude combination (A_+-A_-) and B can be exactly obtained [17, 16]. The computation is rather cumbersome and we present only the final result

$$(A_+-A_-)\xi^2=\frac{1}{8\pi\tan(\pi/N)} \qquad B\xi^2=\frac{1}{8\pi^2} \tag{7}$$

where ξ is proportional to the Compton wavelength of the lightest excitation in the massive regime ($t<0$), which appears as a non-universal scaling factor in the thermodynamic Bethe ansatz formalism [15, 16]. For these theories $c_{\mathrm{IR}}^+=(3N-1)(N-2)/2(2N-1)$ [14]. Therefore, even in the $N\to\infty$, the universal ratio ξ^2/ξ^2 is a well behaved function of N (see (5) and (7)). This is an extra support for our definition of the scale ξ^2.

This result can be simply interpreted in terms of an asymmetric distribution of the complex-t ('temperature') zeros of the partition function for $t>0$ and $t<0$ [11, 12, 13]. The universal angle φ, defining the slope of the singular line of zeros with the (negative) real axis of the complex-t plane is related to the amplitudes $A_\pm$, B by [11, 12]

$$\tan(2\varphi)=-\frac{(A_+-A_-)}{\pi B}. \tag{8}$$

Combining this with (7) we find the following remarkable simple result $2\varphi=\pi/2+\pi/N$. Hopefully, this result will shed some light on the physical interpretation of these $Z(N)$ theories. On the other hand, using (5), the asymmetry on the zeros can be also viewed as a distinct behaviour of the RG trajectories at the infrared fixed point. This suggests that maybe deeper analogies can be established directly from the partition function behaviour away from the critical point.

A second class of interesting theories are those in which one of the amplitudes $A_\pm$ is zero by some symmetry principle. This is the case, for example, of the tricritical Ising model perturbed by the supersymmetry preserving order parameter with conformal dimension $\Delta_\phi=3/5$. Below the fixed point, the off critical theory is believed to be massive ($c_{\mathrm{IR}}^-=0$) and supersymmetric [18, 21, 22] which predicts $f_s^-\equiv 0$. In the opposite direction, however, several results [21, 22, 17] indicate that the theory flow to the Ising fixed point ($c_{\rm IR}^{+}=1/2$) and supersymmetry is spontaneously broken. Hence, in such systems, the formula (5) predicts a non-null $t>0$ amplitude, $f_s^{+}\xi^2 \sim c_{\rm IR}^{+}$, and we may see this result as an indirect mechanism to 'measure' the infrared fixed point $c_{\rm IR}^{+}$. From the definition of the scale ξ^2, however, one has to 'measure' it at the same distance from the critical point in both directions $t \gtrless 0$ [23]. This may bring extra error margins in the scattering experiments, making it not a feasible 'experimental' mechanism. Nevertheless, from the theoretical point of view, our result at least expresses the supersymmetry breaking phenomena in terms of mensurable thermodynamic quantities of the fixed point.

In summary, we have discussed some of the implications of an asymmetric universal amplitude relation associated to the free energy per correlation volume in two-dimensional systems. Examples were discussed in the cases of theories possessing a logarithmic singularity in the free energy and to the crossover from the tricritical to the critical Ising fixed points.

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